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GEOGEBRA ON THE ROCKS

We see great value in making physical
models as mathematical experiments…
(Bryant, 2008)

Introduction

This chapter is about the didactical and mathematical values behind the attempts to build up a GeoGebra model for a 3D-linkage representing a flexible cube, i.e. a cubic framework made up with bars of length (say) one and spherical joints in the vertices. Figure 1 displays two models of the cube: one made with GeoGebra and the other with Geomag[1].

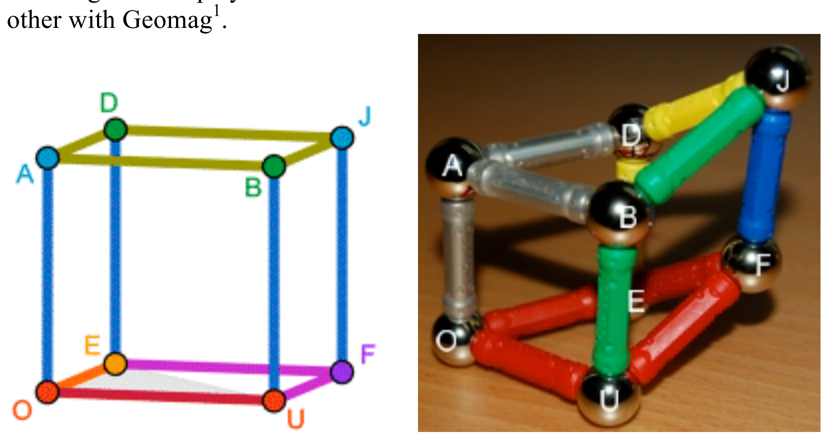

Figure 1. The cube

The importance of making physical models of geometric objects has been widely emphasized (Polo-Blanco, 2007); likewise, we would like to highlight the relevant opportunities that modeling with GeoGebra brings for doing and learning mathematics.

Next section provides arguments in this direction, and introduces the context, main concepts and issues involved in our experiment. Then, a detailed description of the modeling process (and its justification) is provided in a new section. We will like to remark the conjunction of GEOmetry and (computational) alGEBRA that is involved in this process. We end up this Chapter proposing further activities and gathering some Conclusions.

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[1] Geomag is a trademark licensed to Geomag SA.

LINKAGES, DYNAMIC GEOMETRY AND GEOMETRY LEARNING

Linkages

Linkages and mathematics have been, for centuries, closely related topics. A lively account of some issues on this historical relation appears in the recent and wonderful book by J. Bryant and C. Sangwin (Bryant, 2008). Drawing curves (even simple straight lines) with the help of mechanisms is an intriguing topic in which linkages and mathematics meet since the XVIII century. We refer to (Kapovich, 2002) for a modern treatment of these problems, including the proof of a statement conjectured by the Fields medalist W. Thurston on the universality of linkages: "Let M be a smooth compact manifold. Then there is a linkage L whose moduli space is diffeomorphic to a disjoint union of a number of copies of M". It is perhaps remarkable to notice that some work by the Nobel Prize recipient J. Nash, is involved in this proof.

As complementary information, a visit to some web pages, such as those of the *Kinematics Models for Design* Digital Library (KMODDL)[2], at Cornell University or to the *Theatrum Machinarum*[3] of the Universita di Modena, is highly recommended.

Another, but closely related, issue of common interest for mathematicians and engineers is the study of the rigidity (and flexibility) of bar-joint frameworks. As stated in the introduction, in this chapter we will deal with a cube consisting on twelve inextendible, incomprenssible rods (of, say, length one) joined, but freely pivoting, at each of the eight vertices. More generally, we could consider other polyhedral frameworks. An important topic is, then, to decide when the given framework has some internal degrees of freedom (i.e. if it has more possible positions than those that are standard for all rigid bodies in R^3, or in R^2 if we are thinking of planar frameworks).

Famous mathematicians, such as Euler or Cauchy, have worked on diverse versions of this problem, and some conjectures on this context have only been settled in recent times (such as R. Connelly's counterexample to the impossibility of constructing flexible polyhedral surfaces with rigid faces). See (Roth, 1981) for a readable account on this very active field of mathematical research, with applications, for instance, to the design of biomolecules.

Modeling a polyhedral cube as a bar-joint linkage, allows us to experiment with this kind of questions. First of all, if we have in our hands a physical model of a cube framework, it is evident that we can place it around in many different positions, without changing the distances between any pair of its (contiguous or not) vertices. This fact is common to all bodies in three-dimensional space and it is not difficult to verify that there are six parameters governing such displacements, since we can choose an arbitrary position (given by three coordinates in physical space) for one point O on the body, and then we can rotate the body as a whole around this point, with such rotation depending on the so-

2 http://kmoddl.library.cornell.edu/

3 http://www.museo.unimo.it/theatrum/

called three (Euler) angles. Thus we say that all bodies, even rigid ones, enjoy six degrees of freedom in R^3.

Since we are mainly interested in the possible "internal" displacements of the cube (those that change the relative position between some vertices, without breaking the linkage), we would like to discount, once and for all, those six "external" degrees of freedom. Thus, let us assume, as a convention, that we have fixed two contiguous vertices (vertices O and U in Figure 1, left) and that, moreover, vertex E is only allowed to move restricted to a certain plane (for instance, the horizontal plane containing O and U). In this way we are taking care of six displacement parameters: three for fixing vertex O, two for fixing vertex U (since it is constrained to be on a sphere of center O and radius 1) and one for restricting E to be in the intersection of a sphere of center O and radius 1 and in the horizontal plane.

Still, is it possible to move the cube respecting this convention for O, U and E? The answer, obviously, is affirmative (see Figure 1, right) and, thus, we say the cube is non-rigid or that it is flexible. But, how many parameters now rule, respecting this initial setting, the possible displacements of this framework? That is, how many internal degrees of freedom does it have? We will see that this question is highly related to the construction process of a GeoGebra model for our cube: its answer should guide the construction and, conversely, a successful construction should allow experimenting the existence of the different internal displacement parameters.

Dynamic Geometry

In fact, the above circular statement seems just another example of the need of mathematical insight to produce sound dynamic geometry resources, which, on the other hand, help developing mathematical insight into a geometric problem. Yet we think there are some special circumstances in this context.

As it is well known, when opening a Dynamic Geometry worksheet for drawing some sketch there, we are following the traditional paper and pencil paradigm, replacing physical devices (ruler, compass, etc.) by different software tools. The relevant difference is that, in the Dynamic Geometry situation, we can benefit from a dragging feature, which is alien to the paper and pencil context.

Now, bar-joint linkages are physical constructions that include the dragging of some of its elements as an intrinsic feature. No one makes a linkage mechanism to let it stand still. In this sense we could think of Dynamic Geometry programs as specially fit to deal with linkage models. A supporting argument could be a visit to some web pages displaying linkages modeled by dynamic geometry programs; we cannot refrain from suggesting the collection of Cabri-Java applets from one of our co-authors[4], exhibiting an interactive collection of about one hundred mechanisms. Wonderful GeoGebra linkages are displayed at some pages by C. Sangwin[5] or by P. van de Veen[6].

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4 http://jmora7.com/Mecan/mecpral3.htm

5 http://web.mat.bham.ac.uk/C.J.Sangwin/howroundcom/front.html

Modeling bar-joint frameworks through Dynamic Geometry software has some advantages, but also presents some difficulties, compared to the classical case of physical models. In fact, both approaches run smoothly when dealing with very simple polygonal or polyhedral figures. But when it comes to more elaborated items, such as the cube, it is not easy to keep the different pieces assembled, or to avoid collisions between the different bars and vertices (which, in physical reality, tend to be thick, far from being intangible lines and points), if one wants to model some complicated displacements. Our experience with physical models of cubes is that, either they have some relatively large dimensions (and this poses construction problems, for instance, with magnetic forces among different elements) or they tend to be less flexible than expected. Of course, none of these hardships arise with Dynamic Geometry models.

On the other hand, modeling linkages with Dynamic Geometry poses other kind of problems. For instance, it is difficult to model a four-bar planar linkage where all vertices behave similarly (that is, showing in a similar manner the degrees of freedom of the flexible parallelogram when one drags anyone of such vertices). Assume we fix two contiguous vertices (to consider only the internal degrees of freedom), say, O and U.

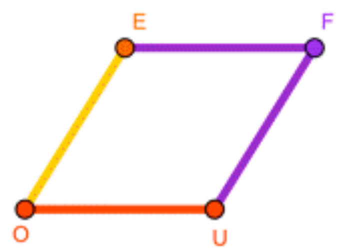

Figure 2. A planar four-bar linkage

Then the two remaining vertices, F, E, should have each one degree of freedom, but not simultaneously. Dragging F, point E should move, and conversely. But a Dynamic Geometry construction tends to assign the shared degree of freedom to just one of them, depending on the construction sequence, and not to the other. Typically, if F is constructed first, we can drag it, then E will move; but we can not drag E… To achieve an homogeneous behavior for E and F we have to use some tricks, such as assigning the degree of freedom to some external parameter and constructing E and F depending on it; or assign the degree of freedom to, say, one single extra point located in the bar joining the two semi-free vertices. It could seem artificial… but we consider that the reasoning required to explain and to circumvent such difficulties is, by all means, an excellent source of geometric thinking.

Last, but not least, we must consider the 3D issue…. Modeling a static 3D object with a Dynamic Geometry program (which has just a 2D display) poses already supplementary problems, not to mention those regarding modeling the

[6] http://www.vandeveen.nl/Wiskunde/Applets%20Constructies.htm

movement of the 3D figure. Doing it, in particular, with GeoGebra, yet without a specific 3D version, is even more challenging. Our experience in this respect is that by using GeoGebra algebraic features we have been able to simulate, reasonably well, 3D scenes and movements for the cube; and to do so much better than using some other Dynamic Geometry program with specific 3D versions we have tried to use in this task.

Geometry Learning

In the previous two sections we have praised the mathematical importance of linkages and the potential role of Dynamic Geometry in modeling such objects. Here we will like to introduce some considerations about the pertinence of introducing linkages as a topic in High School/Undergraduate geometry (Recio, 1998).

In many curricula, movements in the plane are introduced with a certain emphasis in their classification (translations, rotations, symmetries, etc). We can say that movements are considered important for geometry learning, but mostly from a "qualitative" point of view, that is, learning about the different types of rigid movements and their distinctive properties. Now, it is a (mathematically) hard task to classify rigid displacements in the plane, very difficult (for school mathematics) to achieve it for 3D.

Linkages provide a different approach to work with movements in a "quantitative" and intuitive way: how many parameters rule the positions of a point in the plane? And, how many are needed for a triangle? Ditto, for any planar rigid shape? How can we translate this question to the case of a bar-joint framework modeling a triangle, a square, a rectangle, a carpenter rule, etc.? It is easy to reason, at an intuitive level, with such questions, and it is surprising to verify, by direct experimentation with GeoGebra-built linkages, how spatial intuition gets, sometimes, wrong… The case of a bar and joint cube framework is one of these models that provide rich learning situations, and this is one important reason behind our attempts to construct it.

Moreover, simple linkages give rise to complicated (yet classical) high degree curves, by studying the traces of some joints. As documented above, tracing curves through linkages is a lively topic, with many historic anecdotes and relations to technology, a quite appealing topic with lots of classroom activities.

Linkages provide, as well, a good model to understand, through the algebraic translation of the corresponding bar-joint framework construction (see next section for some detailed examples), systems of algebraic equations with an infinite number of (meaningful) solutions. This algebra-geometry dictionary that linkages naturally provide is, in our opinion, one important source of advanced mathematical thinking. And it is particularly close to GeoGebra basic conception, that of mixing Algebra and Geometry in a single tool.

MODELING A CUBE

This section describes the problems and solutions behind our attempts to build a GeoGeobra model of a joint-and-bar cube.

A planar parallelogram

First we analyze the simpler case of a planar joint-and-bar parallelogram (Figure 2) with bars of length one. We might consider fixing vertex O at the origin of coordinates and vertex U at point (1, 0) (in order to focus only on the *internal* degrees of freedom, those that add to the 3 degrees of freedom that, at least, have all planar bodies). Then, counterclockwise, vertex F and vertex E follow. Point F=(Fx, Fy) must be on a circle centered at U and of radius 1. This means only one coordinate of F is free. Finally, point E can be constructed as the intersection of two circles (centered at F and O, respectively) of radius 1. It will have no free coordinates.

In summary, we obtain the following algebraic system:

```
>R:=PolynomialRing([Ex,Ey,Fx,Fy]):
>sys := {(Fx-1)^2+(Fy-0)^2-1, (Ex-Fx)^2+(Ey-Fy)^2-
1,(Ex-0)^2+(Ey-0)^2-1};
```

which can be triangularized, using Maple, as

```
>dec := Triangularize(sys, R): map(Equations, dec, R);

[[Ex-1,  Ey,  Fx^2-2*Fx+Fy^2],  [Ex*Fx-Fx+Fy^2,  Ey-Fy,
Fx^2-2*Fx+Fy^2], [Ex^2+Ey^2-1, Fx, Fy]]
```

We obtain two degenerate solutions (the first and third system in the output above), corresponding to the cases E=U and F=O, and one regular solution, in which Fx is parameterized by Fy; Ey is also parameterized by Fy; and Ex is parameterized by Fx and Fy (thus, by Fy alone). Therefore, algebraically, as well as geometrically, we see the parallelogram has just one internal degree of freedom. But this extra degree of freedom can be assigned to anyone of the coordinates of E or F, depending on the way we order the variables for triangularizing the system or depending on the sequence of the geometric construction.

If we build up a physical joint and bar parallelogram with one fixed side, we observe that we can move any of the two semi-free vertices. Now, no Dynamic Geometry construction seems to achieve this, since the final vertex that is constructed in order to close the loop, has to be determined by the previously constructed vertices; thus only one of the two free vertices would be “draggable”…

A spatial parallelogram

Then we will deal with the slightly more complicated case of a 3D joint-and-bar parallelogram.

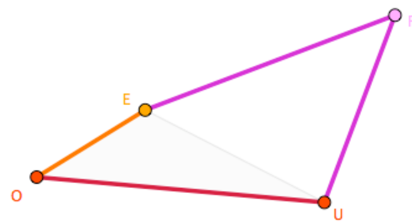

Figure 3. A spatial four-bar linkage

The most evident difficulty for GeoGebra to model this linkage is the lack of 3D facilities. We can circumvent this difficulty thanks to the algebra integrated in GeoGebra. We will associate to each tridimensional point (Px, Py, Pz) its projection (Qx, Qy) on the screen, this projection depending on some user-choice parameters alpha and beta (that represent different user perspectives), as follows:

$$(Qx,\ Qy) = (Px,\ Py,\ Pz)\begin{pmatrix} \sin(\beta) & -\sin(\alpha)\ \cos(\beta) \\ \cos(\beta) & \sin(\alpha)\ \sin(\beta) \\ 0 & \cos(\alpha) \end{pmatrix}$$

Once the user introduces, by clicking on some icon such as the two ellipses of Figure 4,

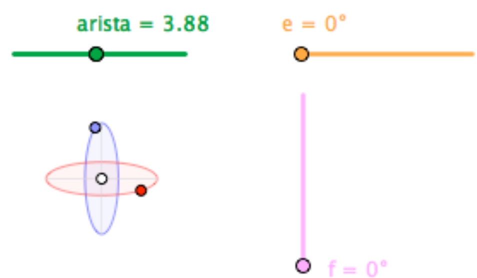

Figure 4. Control icons

the values of alpha and beta, GeoGebra projects on the screen the corresponding values of the different tri-dimensional points that will be introduced through numerical coordinates.

Here we fix two adjacent vertices (say, O = (0,0,0) and U= (0,1,0)) and the plane (of equation z=0) where another vertex (say, E) should lie. In this way we take care of the 6 common degrees of freedom for all 3D shapes. Therefore, the coordinates for E are

$$E = (Ex, Ey, 0)$$

Since E must be at distance 1 from O, these coordinates verify:

$$Ex^2 + Ey^2 = 1$$

That is, introducing a new parameter e:

$$E = (-\cos(e), \sin(e), 0)$$

This parametric representation can be achieved in GeoGebra by constructing a slider (see Figure 4) that will control angle e in order to move point E.

Now, concerning vertex F = (Fx, Fy, Fz), we observe that, being equidistant to E and U, it must be in a plane perpendicular to segment UE through the middle point Q of this segment. But this plane goes also through O, since OE and OU have same length. Therefore, the coordinates of F verify the following system of equations: `{(Ex-0)^2+(Ey-0)^2-1, Ez, (Fx-0)^2+(Fy-(1))^2+(Fz-(0))^2-1,(Fx)*Ex+(Fy)*(Ey-1)+(Fz)*Ez}` and it is not difficult to see that eliminating all variables from this system, except those corresponding to the coordinates of F, one obtains just the sphere `(Fx-0)^2+(Fy-(1))^2+(Fz-(0))^2 =1`.

A more geometric way of arriving to the same result could be the following. We observe that, for fixed E, point F describes a circle centered a Q and of radius equal to k1 (see below for the value of this parameter).

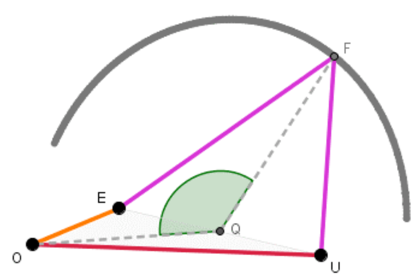

Figure 5. Determining F

Parametrizing by a new angle f the position of F in this circle we get:

$$Fx = -\cos(e)/2 - \operatorname{sgn}(\cos(e))\ k_1\ \cos(f)\ \sin(k_2)$$

$$Fy = (\sin(e)+1)/2 + k_1\ \cos(f)\ \cos(k_2)$$

$$Fz = k_1\ \sin(f)$$

where k_1 y k_2 are given by:

$$k_1 = \text{sqrt}(2 + 2\sin(e))/2$$

$$k_2 = \text{acos}(\sin(e))/2$$

Thus we remark that there are, in total, two internal degrees of freedom (angles e and f), which are distributed between the two free vertices (one for each vertex), in the following sense: E moves on a circle and, for each position of E, F can be placed at whatever point of another circle (with center and radius depending on E's position). From this description it is easy to deduce that the locus of all possible placements of F is a surface parameterized by circles of variable radius, centered at the different points of the circle displayed by the midpoint of EU. After a moment's thought we check that such surface it is just the sphere centered at U, of radius 1, as expected.

The Cube

By considering the case of the spatial parallelogram as a basic building block, we can construct the cube by, first, adding to the parallelogram OUFE a new vertex A with two degrees of freedom (that is, lying on a sphere of given radius and centered at the fixed vertex O), represented by two parameters a and j. Parameter a allows the rotation of A around O with Ax constant; and the parameter j does the same, with Ay constant, that is:

$$A = (Ax, Ay, Az)$$
$$= (\sin(j)\cos(a), \sin(a), \cos(j)\cos(a))$$

Next, from this vertex A, two other adjacent vertices B and D are constructed following the same steps as in the spatial parallelogram case. First, we determine D as the fourth vertex of the parallelogram OAED. Following the arguments of the

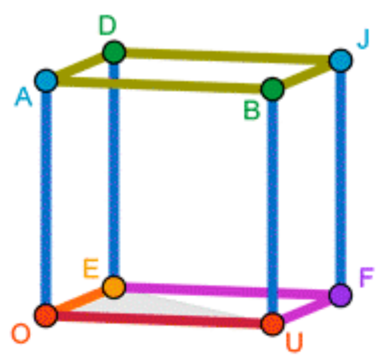

Figure 6. The cube

previous section, for each position of E and A, point D will be parametrized by an angle d on a circle centered at the middle point M of segment AE

```
M =(Mx, My, Mz) = (E+A)/2
```

Moreover D lies on a plane perpendicular to AE and containing O. Thus

```
OD = OM + cos(d) OM + sin(d) |OM| n/|n|
```

where n is the vector product of OM by EM,

```
n = (Mz Ey, - Mz Ex, Mx(My - Ey) - My(Mx - Ex))
```

which is perpendicular to OD and to EA.

Likewise, we can determine now (that is, as the fourth vertex of parallelogram OUBA, assuming O, U and A are fixed) vertex B depending on a new parameter b:

```
N =(Nx, Ny, Nz) = (U+A)/2
m = (Nz, 0, -Nx)
OB = ON + cos(b) ON + sin(b) |ON| m/|m|
```

where N is the midpoint of UA and m is the vector product of ON by UN.

It remains to parametrize vertex J. We observe that, for given positions of O, U, E, F, A, B, D, this vertex must be on the intersection of three spheres of same radius, centered at F, B and D. Therefore, there are, at most, two possible (isomer) positions for J = (Jx, Jy, Jz). We obtain their coordinates by considering that EJ (and UJ) must be perpendicular to DF (to BF, respectively):

```
(Jx-Ex)(Dx-Fx) + (Jy-Ey)(Dy-Fy) + Jz(Dz-Fz) = 0
Jx(Bx-Fx) + (Jy-1)(By-Fy) + Jz(Bz-Fz) = 0
```

The intersection of these two planes (remark that only the J-coordinates are unknown here) will be a line in the direction determined by the vector product of the normal vectors to these two planes. Finally we look for the intersection points of this line with the sphere centered at F and or radius 1:

```
(Jx-Fx)^2 + (Jy-Fy)^2 + (Jz-Fz)^2 = 1
```

yielding the two possible positions of J. The resulting expression is too large to be reproduced here.

Figure 7 above displays the cube for some given, through the sliders on the top of the Figure, values of the parameters we have introduced in this section. The same values, for another isomer position of J, yield the cube at the position displayed on Figure 7, below.

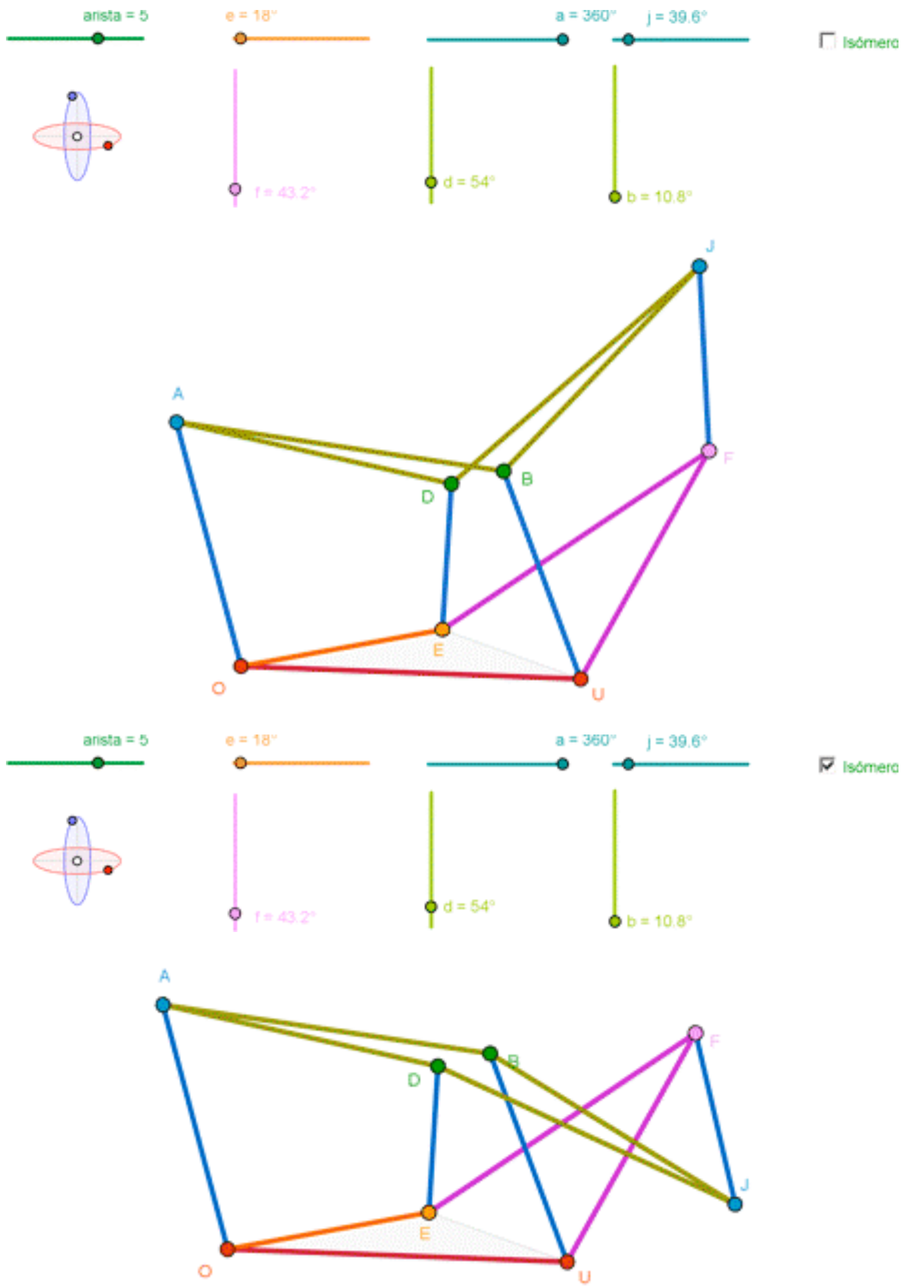

Figure 7. Two isomers for same value of parameters

OPEN ISSUES AND CONCLUSIONS

The construction of the cube model that we have described in the previous sections behaves quite well in practice. Setting the slides at different positions, the different vertices of the cube —GeoGebra numerically computes their coordinates following the corresponding parametrization and then projects them onto the screen by performing some more arithmetical operations—are instantaneously placed at the expected positions. Yet, we have to report that some "jumps" occur between isomer positions, near singular placements (for instance, when a=270º and parallelogram AOBU collapses). In view of the large bibliography on the "continuity problem" for Dynamic Geometry, it seems a non-trivial task to model a cube avoiding –if possible at all-- such behavior.

We remark that the cube we have modeled has six internal degrees of freedom, one for each free parameter we have introduced. But its distribution has not been homogeneous. For instance, the final vertex has been constructed without any degrees of freedom, by imposing some constraints (being simultaneously in a sphere and in two planes perpendicular to some diagonals). This fact –the difficulty to make a model where all semi-free vertices behave homogeneously—is apparently similar to the planar parallelogram case, but now we can not conclude that it is impossible to make such construction, since, after fixing O and U we still have six vertices and six degrees of freedom. It is, probably, a consequence of our approach and not an intrinsic characteristic.

In fact, we can think of the Dynamic Geometry sequential construction process as a kind of triangularization of the system describing a cube. In the planar parallelogram case the triangularization of the system yield always one semi-free vertex depending on the other one. In principle for a cube, a triangularization should be possible with one new free variable associated to each semi-free vertice, but the triangularization (or Gröbner basis computation) of the algebraic system describing the distance 1 constraints between some pairs of vertices of the cube seems not feasible (due to the complexity of the involved computations). If it would have succeeded computing automatically this general solution we could have shown automatically that, in fact, the cube has six (internal) degrees of freedom. Right now this important fact can be just proved by considering the specific sequence of solutions presented in our construction, depending on six parameters. In some sense, we see that attempting to build a model of a cube is an example where GeoGebra helps when symbolic computation fails. And, conversely, it shows how symbolic computation (for 3D coordinates) helps when current GeoGebra features fail.

Building a cube with GeoGebra provides excellent opportunities to learn a lot of mathematics at different levels. Some of them have been summarily introduced in the construction process (such as discussing why the intersection of three spheres has at most two points, or why vertex F in a spatial parallelogram moves on a sphere, etc.). Not to speak about the interaction of algebra (dimension of the algebraic variety defined by the cube's equations, triangular systems, etc.) and geometry that is behind our construction.

Moreover, different classroom exploration situations can be presented to work and play with the GeoGebra cube model, such as:

-Could you fix (say, by pasting some rigid plates) one, two… etc. facets in the cube and still have some flexibility on the cube? How many internal degrees of freedom will remain?

-For a planar parallelogram, one can feel the one-degree of freedom by checking that once you fix one semi-free vertex, the whole parallelogram gets fixed. Same for the spatial parallelogram (you have got to fix, one after another, the two semi-free vertices). For the cube, how can you "feel" its six degrees of freedom? Can you fix whatever five semi-free vertices and still move the cube?

The cube, its construction process and the model itself, seems to us an important source of both algebraic and geometric insight. And, most important, an endless source of fun thanks, as always, to GeoGebra.

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